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A SIMPLE PURSUIT-AND-EVASION GAME ON A TWO-DIMENSIONAL CONE†

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A differential pursuit-and-evasion game is considered in which the players—velocity-controlled points in three-dimensional Euclidean space—move on a two-dimensional conical surface, i.e. at each instant of time the players may choose their velocity vectors in an arbitrary direction along the tangent of the cone (the magnitude of the velocity vectors is bounded by a constant). The pursuer has a strict velocity advantage. It is shown that self-similar variables reduce the original game with dynamic equations of fourth order to a two-dimensional game. The necessary conditions of optimality are applied to construct a complete solution of the positional pursuit-and-evasion game. It is shown that in the main part of the phase space the optimal motion of the players is along the connecting geodesics. In the other part of the space, each player moves along his own geodesic; the envelopes of these geodesics are singular equivocal trajectories. The equivocal surface is a basic element of synthesis, enabling a complete optimal phase portrait of the game to be constructed. A third kind of motion is obtained for certain parameter values. In the corresponding subregion of the phase space, the pursuit time is independent of the evader position; starting from any point, the players meet at the vertex of the cone.

Sufficiency of the optimality conditions is not considered. The present paper uses the methods described in [1] and develops its results.

1. STATEMENT OF THE PROBLEM

SUPPOSE that the points (players) P and E move on a closed two-dimensional conical surface K_0 in three-dimensional Euclidean space (Fig. 1). The velocity of the point P does not exceed 1 and the velocity of the point E does not exceed ν, $0<\nu<1$. The open set $K_0\backslash 0$—a cone without the vertex 0—will be denoted by K. Parametrizing the cone K, we represent the equations of motion of the points P and E on the two-dimensional surface in the form

$$P : x^{\cdot} = u, \ u \in E_1(x), \ E : y^{\cdot} = v, \ v \in E_\nu(y) \tag{1.1}$$

Here $x, y \in R^2$ are the local coordinates of the points P and E; $E_\alpha(x) \equiv \{u \in R^2: \langle G(x)u, u\rangle \leqslant \alpha^2\}$ is the ellipse of tangent vectors at the point $x \in K$, $\alpha \geqslant 0$; $G(x)$ is the metric tensor of the manifold K. We assume that the metric G is induced on K by the Euclidean metric of the embedding three-dimensional space. The positive definite matrix $G(x)$ defines the first quadratic form of the surface [2]; its elements can be computed by expressing the square of the differential of the Euclidean length of an arc on K in terms of the parameters x.

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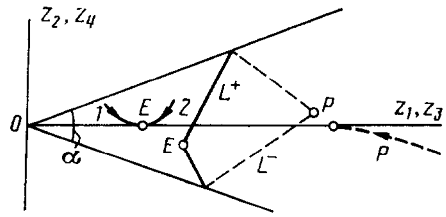

FIG. 2.

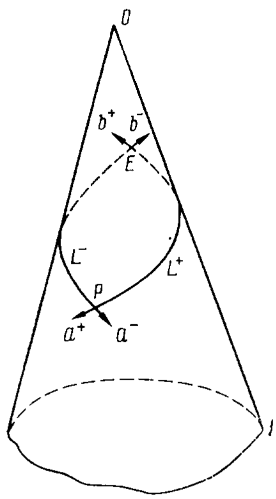

FIG. 1.

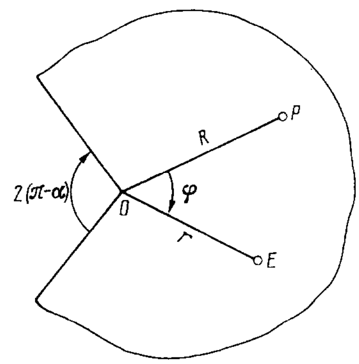

FIG. 3.

Denote by $L = L(x, y)$ the distance between the points P, $E \in K$, i.e. the length of the minimal geodesic $\gamma \subset K$ connecting P and E. The game starts at time $t = 0$ and ends at time $t = T > 0$, when we first have

$$L\ (x\ (T),\ y\ (T)) = 0 \tag{1.2}$$

Player P attempts to minimize the time T and E attempts to maximize it. For $t = 0$, we have $l > 0$.

The game (1.1) and (1.2) will be considered in the class of rotational controls, assuming that the players have complete information about the game dynamics and the current position $(x, y) \in K \times K$ [3].

Since the cone K can be unfolded on the plane, the local coordinates may be chosen to be Euclidean, i.e. there exists a change of variables $x, y \rightarrow x, y$ such that the equation of simple motion (1.1) take the form

$$P : x^{\cdot} = u,\ |\ u\ | \leqslant 1,\ E : y^{\cdot} = v,\ |\ v\ | \leqslant \nu \tag{1.3}$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$ are the local coordinates of the points P and E (Fig. 2) and $|u|^2 = u_1{}^2 + u_2{}^2$ is the Euclidean length of the velocity vector u. We also introduce the four-dimensional phase state vector of the game $z \in Q^4$: $z_i = x_i$, $z_{i+2} = y_i$, $i = 1, 2$. This change of variable arises, for instance, when the cone is subjected to the following deformation: it is folded along any pair of opposite generators $\gamma_1, \gamma_2 \subset K$ into a plane two-sided angle, which is also denoted by K. The length of the geodesic is preserved under this transformation (Fig. 2). We denote by α in Figs 2 and 3 the half-angle of the cone when fully unfolded on the plane, $0 < \alpha < \pi$. For $\alpha = 0$, K_0 degenerates into a ray; for $\alpha = \pi$, it is unfolded into the entire Euclidean plane. On the angle K, the geodesics are polygonal lines with equal angles of "incidence and reflection" relative to the sides of the angle K (Fig. 2).

We may thus identify the physical space of the game with the plane two-sided angle K_0: the game ends if the points P and E are on the same side of the angle and their Euclidean coordinates are equal.

The players, located on the sides γ_1 and γ_2 of the angle K, may continue their motion both on the "direct" and on the "reverse" side of the angle. Strictly speaking, the components of the vectors x, y should be equipped with an additional two-digit index that identifies the side of the surface on which the points P and E are located. For simplicity, this index is omitted and some additional verbal explanation is added when needed. Note that different choices of the generators γ_1 and γ_2 for folding the cone into a plane angle corresponds to different partitions of the manifold K into charts [2].

The players may also be located at the vertex O of the cone K_0. There is no tangent plane to K_0 at the point O, and therefore the constraints on the velocities of the players are not described by inclusions of the form (1.1). A tangential cone exists at the point O (it is identical with K). Admissible velocities for $P = O (E = O)$ are vectors in the three-dimensional Euclidean space directed along the generators of the cone K and not exceeding 1 (ν) in absolute value. A more detailed analysis of the singularity associated with the vertex O is not required, because, as we shall see, with optimal behaviour of the players the vertex may only act as the initial or the final point of the trajectories. In other words, the smooth manifold K may also be considered as the phase manifold of the game, with condition (1.2) replaced in case of capture at the point O by the limit relationship $L \to 0$ as $t \to T-0$.

The construction of positional controls and stopping conditions only requires a knowledge of the relative positions of the players. Therefore, using the variables r, R, φ, whose meaning is clear from Fig. 3, we can restate the game (1.1)–(1.3) in the following form in terms of three-dimensional equations of dynamics and the stopping condition

$$\begin{gathered} R^{\cdot} = u_1, \quad r^{\cdot} = v_1, \quad \varphi^{\cdot} = v_2/r - u_2/R, \quad |u| \leqslant 1, \quad |v| \leqslant \nu \\ R(T) = r(T), \quad \varphi(T) = 0 \quad (R(T) = r(T) = 0) \end{gathered} \tag{1.4}$$

Here u_i and v_i are the projections of the velocities of the points P and E on the axes of moving rectangular coordinate systems; the equality in parentheses corresponds to the encounter of the players at the vertex O, when the value of the angle $\varphi(T)$ is undefined. The variables r, R, φ: $0 \leqslant r$, $R < \infty$, $|\varphi| \leqslant \alpha$ are uniquely expressible in terms of the local coordinates x, y used in (1.3)

2. SELF-SIMILAR VARIABLES. THE TWO-DIMENSIONAL GAME PROBLEM

Consider the transformation of the cone K to itself, applying contraction by a factor λ, $\lambda > 0$, along all its generators. The unfolded plane figure shown in Fig. 3 will be similarly contracted. The points P, E go to some points P_λ, E_λ, and the length of any trajectory traversed by the players as well as the time to traverse the trajectory (with initial velocities) are reduced by a factor λ. Therefore, contraction by a factor $\lambda = 1/R$ may reduce the analysis of the game with any initial position (r, R, φ) to the analysis of a standard position (ρ, I, φ), where $\rho = r/R$. This transformation is allowed because the problem does not have a characteristic length; in the pursuit-and-evasion game in a plane with an obstacle [4], for instance, this contraction alters the size of the obstacle and self-similarity does not hold.

Using this argument, we can show that the complete system of relationships that describe the game, including the Bellman equation, is invariant with respect with the one-parameter group of contraction transformations. A more direct approach is by formulating an equivalent game in terms of the variables ρ, φ: $0 \leqslant \rho < \infty$, $|\varphi| \leqslant \alpha$. The variables ρ, φ in this region can be used to parametrize the cone K, identifying the pairs $(\rho, 0)$ and $(\rho, 2\alpha)$. A point of the cone is defined by the angle φ on the unfolded figure, measured from some fixed generator, and by the distance ρ from the vertex O. Thus, the mappings $(x, y) \to (r, R, \varphi)$ define the mapping $K \times K \to K$. Having described the game autonomously in terms of (ρ, φ), we thus obtain a situation in which the relative dynamics of two points on K is described by the motion of one point on K. A similar dimension-reducing technique is widely used in cases when the physical space of the game is Euclidean, see e.g. [3].

We introduce a new time τ related to the original time t by the differential relationships $dt/d\tau = R$. Differentiating the equality $\rho = r/R$ with respect to τ (this operation is denoted by a prime) and using (1.4), we obtain the equations of motion in the form

$$\rho' = v_1 - \rho u_1,\ \varphi' = v_2/\rho - u_2,\ |u| \leqslant 1,\ |v| \leqslant \nu \tag{2.1}$$

Note that the mapping $t \to \tau$ depends on the particular realization of the function $R(t)$, $0 \leqslant t \leqslant T$, i.e. on the choice of the control $u_1(t)$ [see (1.4)]. We assume that the time $\tau = 0$ corresponds to $t = 0$.

To represent the functional of the game, we require the dependence $R = R(\tau) = R_0(\tau)$, $\tau \geqslant 0$, which is obtained by integrating the relationship $R' = Ru_1$ that follows from (1.4) and determining the time τ. We have

$$\begin{gathered} T = \int_0^T dt = \int_0^\theta R(\tau)\,d\tau = R_0 \int_0^\theta I(\tau)\,d\tau \\ I(\tau) = \exp\left(\int_0^\tau u_1(\mu)\,d\mu\right)\quad R_0 = R(0) \end{gathered} \tag{2.2}$$

Omitting the positive multiplier R_0, we can represent the functional of the game in the variables ρ, φ, γ in the form

$$J = \int_0^\theta I(\tau)\,d\tau \tag{2.3}$$

The functional (2.3) does not have the additivity property, which is responsible, for instance, for the applicability of the maximum principle in optimal control problems. The dynamic programming approach leads to the Bellman equation, which contains both the gradient of the Bellman function and the Bellman function itself.

In the variables ρ, φ, τ, the stopping conditions are somewhat more complicated. If, for instance, the players meet at the vertex of the cone $R = 0$, then from the dependence $R(\tau)$ we see that the interval $[0, T]$ is mapped in the infinite interval $[0, \infty)$ and the functional (2.3) remains finite (equal to T/R_0). Thus, capture may occur in a finite or infinite time $\tau = \theta$ when the following conditions are satisfied:

$$\begin{gathered} \theta < \infty;\ \rho(\theta) = 1,\ \varphi(\theta) = 0 \\ \theta = \infty;\ \rho_0 + \int_0^\theta v_1(\xi) I(\xi)\,d\xi = 0,\quad \int_0^\infty u_1(\mu)\,d\mu = -\infty \end{gathered} \tag{2.4}$$

The inequalities in the second case follow from the conditions $r \to 0$, $R \to 0$ as $\tau \to \infty$. The encounter of the players in this case may occur at any point in the space ρ, φ, including $\rho = \infty$, because the mapping $(r, R, \varphi) \to (\rho, \varphi)$ has a singularity at the point $O = (0, 0, \varphi)$ and its image (in the limiting sense) may be any point of the cone K. This is not a fundamental difficulty, first, because of the special role of the point O in the context of optimal synthesis and, second, because the variables (ρ, φ) will be used mainly to represent the results, which are constructed using the variables (1.3) and (1.4).

Note that the control parameters u_i, v_i in (1.3) and (1.4) are the projections of the velocities, u, v of the points P, E on different rectangular coordinate systems, although the same notation is used. Thus, the dynamics of the game is represented in (1.3), (1.4), and (2.1) in terms of three coordinate systems, which will be called Cartesian, relative, and self-similar variables, respectively. Regions and surfaces (curves) in the space ρ, φ will be denoted for simplicity by the same symbols as their sources under the mapping $(x, y) \to (\rho, \varphi)$.

3. THE NECESSARY CONDITIONS OF OPTIMALITY

Denote by V the value of the game (the Bellman function), treating it as a function of the arguments $z(x_1, x_2, x_3, x_4)$, $z \in R^4$, or r, R, φ. If the function V is directionally differentiable, then the generalized necessary conditions of optimality are written in the form [1, 5]

$$\min_u \max_v V^{\cdot} \geqslant -1 \geqslant \max_v \min_u V^{\cdot}, \quad |u| \leqslant 1, \quad |v| \leqslant \nu \tag{3.1}$$

Here $V^{\cdot}$ is the total derivative with respect to time by Eqs (1.3) or (1.4), i.e. the directional derivative of the vector of the right-hand sides. At points of differentiability of the function V, (3.1) reduce to equalities, which define the Bellman equation in Cartesian and relative variables, respectively.

$$\begin{gathered} F(p) \equiv -\sqrt{p_1^2 + p_2^2} + \nu \sqrt{p_3^2 + p_4^2} = -1 \quad (p = V_z \in R^4) \\ -\sqrt{V_R^2 + V_\varphi^2/R^2} + \nu \sqrt{V_r^2 + V_\varphi^2/r^2} = -1 \end{gathered} \tag{3.2}$$

Denote by $Q = Q(\rho, \varphi)$ the optical optimal outcome function in problem (2.1), (2.3). By (2.2), we have the identity $V(r, R, \varphi) = RQ(r/R, \varphi)$. Differentiating, we obtain

$$V_r = Q_\rho, \quad V_\rho = Q - \rho Q_\rho, \quad V_\varphi = RQ_\varphi \ (\rho = r/R)$$

Using this equality in the second relationship in (3.2), we obtain the Bellman equation for problem (2.1) and (2.3)

$$\Phi = -\sqrt{Q_\varphi^2 + (Q - \rho Q_\rho)^2} + \nu \sqrt{Q_\rho^2 + Q_\rho^2/\rho^2} + 1 = 0 \tag{3.3}$$

Equation (3.3) containing the required function Q can also be obtained by applying the dynamic programming approach directly to the game (2.1) and (2.3).

The boundary conditions for the required functions in Eqs (3.2) and (3.3) are respectively

$$V(x, x) = 0, \quad V(r, r, 0) = 0, \quad Q(1, 0) = 0 \tag{3.4}$$

These boundary conditions indicate that the pursuit time is zero if the points P and E coincide, i.e. if condition (1.2) holds.

The Bellman equations (3.2), (3.3) are nonlinear first-order partial differential equations of the form $F(z, V_z(z)) = 0$, $z \in Z \subset R^n$. They generate a system of ordinary differential equations of order $2n+1$ (the characteristic system) [6]

$$z^{\cdot} = F_p, \quad p^{\cdot} = -F_z - pF_V, \quad V^{\cdot} = \langle p, F_p \rangle \quad (p = V_z) \tag{3.5}$$

which is used for the local construction of the required function $V(z)$. The function F for equations of the form (3.2) does not depend explicitly on V, and therefore the last equation in (3.5) is separated from the Hamiltonian system $z^{\cdot} = F_p$, $p^{\cdot} = -F_z$.

The optimal trajectories of the players in regions where the conditions of regularity are satisfied [twice smoothness of the functions V, F, uniqueness of the extrema (3.1)] are determined by the characteristic equations. In Cartesian variables, using the function F in (3.2), we write the Hamiltonian equations in the form

$$z_i^{\cdot} = F_{p_i} \equiv -\frac{p_i}{\sqrt{p_1^2 + p_2^2}}, \quad z_{i+2}^{\cdot} = F_{p_{i+2}} \equiv \frac{\nu p_{i+2}}{\sqrt{p_3^2 + p_4^2}} \ (i = 1,2), \quad p^{\cdot} = F_z \equiv 0 \tag{3.6}$$

The components of the vector F_p in (3.6) determine the optimal controls u, v in (1.3) depending on the conjugate vector p; the extrema in (3.1) are attained on these controls. Substituting $p = V_z(z)$ in the vector F_p, we obtain the optimal positional controls of the players $u(z)$, $v(z)$. System (3.6) is of eighth order, the Hamiltonian equations in relative variables are of sixth order, and the complete system of characteristic equations in self-similar variables is of fifth order

$$\rho^{\cdot} = \Phi_\xi, \quad \varphi^{\cdot} = \Phi_\eta, \quad \xi^{\cdot} = -\Phi_\rho - \xi\Phi_Q, \quad \eta^{\cdot} = -\Phi_\varphi - \eta\Phi_Q, \quad Q^{\cdot} = \xi\Phi_\xi + \eta\Phi_\eta \tag{3.7}$$

Here $\xi = Q_\rho$, $\eta = Q_\varphi$; the function $\Phi = \Phi(\rho, \varphi, \xi, \eta, Q)$ is defined in (3.3).

4. PRIMARY SOLUTION

In some region $Z_1 \subset K \times K$ of the phase manifold of the game, the optimal behaviour of the players is pursuit and evasion with maximum velocities along the minimal geodesic connecting the points P and E (the game is conducted on a non-Euclidean plane). This assertion can be proved independently [1] and it also follows from our optimal synthesis in the entire space. The pursuit time in this motion (the value of the game) is

$$S(z) = L(z)/(1 - \nu), \quad L(z) = \min\,[L^+(z),\ L^-(z)] \tag{4.1}$$

Here L is the length of the minimal geodesic, L^+, L^- are the local length minima corresponding to two possible motions from point P to E (Fig. 2). In the Cartesian variables, L^+ and L^- are defined by the equalities

$$L^\pm = [\,|x|^2 + |y|^2 - 2\cos\alpha\,(x_1y_1 - x_2y_2) \mp 2\sin\alpha\,(x_2y_1 + x_1y_2)]^{1/2} \tag{4.2}$$

which are obtained by solving the problem of the minimum-length two-link polygonal line connecting the points P and E with a break on the ray γ_1 and γ_2. Thus, $S(z) = V(z)$ for $z \in Z_1$.

It is shown in [1] that the time (4.1) is guaranteed for player P starting from any position, i.e. $V(z) \leq S(z)$, $z \in Z = K \times K$.

The function (4.1) satisfies the boundary condition (3.4) [see (1.2)] and Eq. (3.2) at the points $z \in Z_1$, where $L^+(z) \neq L^-(z)$, i.e. the function $S(z)$ is differentiable. The last property is ensured by the fact that the functions (4.2) satisfy the eikonal equation in the first and second pair of arguments [1]

$$L_{x_1}^2 + L_{x_2}^2 = 1, \quad L_{y_1}^2 + L_{y_2}^2 = 1.$$

In self-similar variables, the primary solution of Eq. (3.3) has the form

$$W(\rho, \varphi) = h(\rho, \varphi)/(1 - \nu), \quad h(r/R, \varphi) = L(r/R, 1, \varphi) = L(r, R, \varphi)/R, \quad h = \min\,[h^+, h^-], \quad h^\pm = \sqrt{1 + \rho^2 - 2\rho\cos(\varphi - \alpha \pm \alpha)} \tag{4.3}$$

i.e. $W = Q$ for $(\rho, \varphi) \in Z_1$ (the symbol Z_1 is also used for the primary region in the variables ρ, φ). Here $L(r, R, \varphi)$ is the length of the minimal geodesic as a function of the relative coordinates.

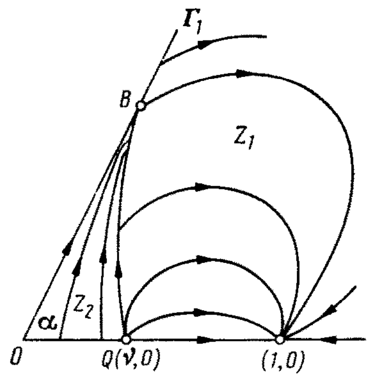

FIG. 4.

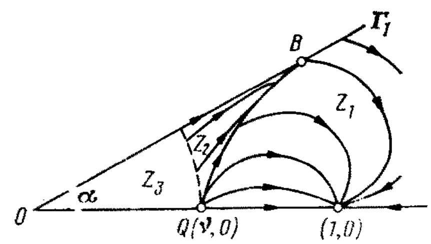

FIG. 5.

The optimal trajectories of the players on the two-sided angle in the primary region are polygonal lines. They are shown in Figs 4 and 5 in the polar coordinates ρ, φ. Because of symmetry, we only show half the phase space corresponding to $0 \leqslant \varphi \leqslant \alpha$; the other half of the picture is a mirror image relative to the ray OB. The family of primary trajectories in the coordinates ρ, φ is defined by the equalities

$$\rho^2 = \frac{1+\nu^2\sigma^2 - 2\nu\sigma\cos\mu}{1+\sigma^2-2\sigma\cos\mu}, \quad \operatorname{tg}\varphi = \frac{\sigma(1-\nu)\sin\mu}{1+\nu^2\sigma^2-\sigma(1+\nu)\cos\mu} \tag{4.4}$$
$$0 \leqslant \mu \leqslant \pi, \quad \sigma \geqslant 0$$

The parameter μ identifying a curve of the family (4.4) equals the angle between the trajectory (the geodesic PE) and the generator of the cone passing through the point of encounter. Each curve is parametrized by the variable $\sigma \geqslant 0$, the value $\sigma = 0$ corresponds to capture: $\rho = 1$, $\varphi = 0$.

Formulas (4.4) can be obtained by considering in the variables r, R, φ the rectilinear planar motion of the players P and E with maximum velocities along the connecting ray toward the point of encounter M, using the parameter $\sigma = (T-t)/|M|$, where $|OM|$ is the distance from the vertex of the cone to the point of encounter.

Also note that the family (4.4) consists of the integral curves (reaching the point $\rho = 1$, $\varphi = 0$) of the system of thr first two equations in (3.7) with $\xi = W_\rho$, $\eta = W_\varphi$, where W is defined in (4.3).

The ray $\varphi = \alpha$ originating from some point B (Figs 4 and 5) is the scattering line, corresponding to location of the points P and E on opposite generators of the cone. The scattering curve is determined by the conditions $h^+ = h^-$. The ray $\varphi = 0$ includes two trajectories that reach the terminal point $\rho = 1$; on the original cone, these trajectories correspond to the motion of the players along a common generator toward the vertex or away from it.

The point B and the singular equivocal trajectory reaching this point are constructed by the method described in [1].

5. SINGULAR EQUIVOCAL MOTION

Denote by Γ_1 the part of the scattering surface $L^+(z) = L^-(z)$ which lies in the primary region Z_1, i.e. is an element of optimal synthesis. It has been proved [1] that Γ_1 is defined by the condition

$$\Gamma_1\colon L^+(z) = L^-(z), \quad F(R_z(z)) \geqslant 0,$$
$$R(z) = (L^+(z) + L^-(z))/(2(1-\nu)) \tag{5.1}$$

Inequality (5.1) follows from the left inequality in (3.1) substituting $V^{\bullet} = \min[L^{+\bullet}, L^{-\bullet}]/(1-\nu)$ [see (4.1)]. On the set (5.1), the left condition in (3.1) (min-max-min) is satisfied as equality, which indicates that the player P has a pure positional strategy [3, 5] on Γ_1. The right condition in (3.1) (max-min-max) is satisfied on the entire surface $L^+(z) = L^-(z)$ as a strict inequality (player E does not have a pure positional control).

The edge B of the surface (5.1) is defined by two equalities

$$B: L^+(z) = L^-(z), \quad F(R_z(z)) = 0 \tag{5.2}$$

The second equality can also be written in the form [1]

$$|a^+ + a^-| - \nu |b^+ + b^-| = 2(1-\nu) \; (a = L_x, \, b = L_y) \tag{5.2'}$$

where $a^{\pm}(z)$ and $b^{\pm}(z)$ are the outer unit tangent vectors to the two geodesics of equal length L^+, L^- joining the points P and E (Fig. 1). The equalities in parentheses are satisfied because of the eikonal equations mentioned previously.

The set B (5.2) is two-dimensional in the four-dimensional z-space. In self-similar variables, B is zero-dimensional, i.e. it is a point with coordinates $\rho = \rho_B$, $\varphi = \alpha$.

The equation for ρ_B may be derived geometrically, using the representation (5.2). A simpler technique is to substitute into the function Φ (3.3) half the sum of the primary solutions (4.3), similar to (5.1); then the equality $\Phi = 0$ defining the edge of the manifold (5.1) in the variables ρ, φ takes the form

$$\rho\,|1 + h^2 - \rho^2| - \nu\,|\rho^2 + h^2 - 1| = 2(1-\nu)\,\rho h; \quad h = \sqrt{1 + \rho^2 - 2\rho\cos\alpha} \tag{5.3}$$

In the plane of the parameters α, ν (Fig. 6), the root ρ_B of Eq. (5.3) in the regions Π_1, Π_2 is defined respectively by the equalities

$$\rho_B = [\nu(1 \pm \cos\alpha) \mp (1-\nu)\sqrt{2\nu(1 \mp \cos\alpha)}]/[\cos\alpha \pm (2\nu - 1)] \tag{5.4}$$

On the critical curve $\Pi_* = \{(\alpha, \nu): \nu = 1 - \sin\alpha, 0 < \alpha < \pi/2\}$, separating the regions Π_1, Π_2 both roots are equal $\rho_B = \cos\alpha$.

The existence of a unique root (5.4) of Eq. (5.3) indicates that to each point $E \in K$ at a distance $r > 0$ from the vertex (Fig. 3) there corresponds a unique point $P \in K$ on the opposite generator at a distance $R = r/\rho_B$ from the vertex and such that the pair P, E is contained in the set $B \subset K \times K$. This correspondence defines a smooth mapping $K \to B$; in other words, in Cartesian variables, the set B is a smooth two-dimensional manifold diffeomorphic to the cone K.

Further constructions rely on the assumption that the edge B of the surface Γ_1 is also the edge of another singular surface Γ_2; the equivocal surface consisting of two branches Γ^+, Γ^- [1, 7], in accordance with the qualitative picture in Fig. 7.

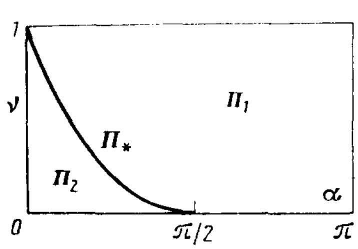

FIG. 6.

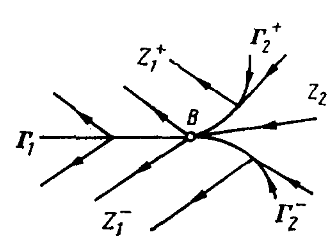

FIG. 7.

The equivocal surface is the switching (discontinuity) surface of the optimal controls of both players and it consists of singular optimal motions [7]. In these motions, on reaching the surface one of the players (the designated player for the given surface) does not switch and continues using the "old" control. In the case of simple motions, when the extrema in (3.1) for the points of smoothness of the function $V(z)$ are attained on unique vectors, the optimal trajectories are necessarily tangent to the equivocal surface. Three necessary conditions of optimality are satisfied on the equivocal surface in inequality form: the Bellman equation, the tangency condition, and the continuity condition:

$$F(p) = 0, \ \langle F_p(p), p - q(z)\rangle = 0, \ V - S(z) = 0 \tag{5.5}$$

Here S, $q = S_z$ is the primary solution, V, $p = V_z$ are the value and its gradient in the secondary region Z_2.

The conditions (5.5) uniquely define the system of equations for singular trajectories on the equivocal surface. The construction procedure is described in [7]. In reverse time, which is useful for backward constructions, these equations together with the initial conditions have the form [4, 7]

$$z^{\cdot} = -F_p, \ p^{\cdot} = -[\langle S_{zz}^{\pm}F_p, F_p\rangle / \langle F_{pp}q^{\pm}, q^{\pm}\rangle](p - q^{\pm}) \tag{5.6}$$

$$z(0) = z^{\circ}, \ p(0) = R_z(z^{\circ}), \ z^{\circ} \in B$$

Here S_{zz} and F_{pp} are the symmetric matrices of second partial derivatives of the functions defined in (4.1) and (3.2). The function $R(z)$ is defined in (5.1). For the branches $\Gamma^{\pm}$ we use the primary solutions $S^{\pm}(z) = L^{\pm}(z)/(1-\nu)$ in (5.5) and (5.6).

The initial values of the conjugate variable at the points of the manifold B, shown in (5.6), were obtained by solving the following system of four equations:

$$F(p) = 0, \ \langle F_p(p), q^{\pm}(z)\rangle + 1 = 0, \ \langle p - q^{\pm}, r_j\rangle = 0, \ j = 1, 2 \tag{5.7}$$

Here $r_j(z) \in R^4$ are two linearly independent vectors tangent to the manifold B at the point $z \in B$. The last two equations in (4.6) were obtained by differentiating the left-hand side of the last equality in (5.5) with respect to the directions r_j. The second equality in (5.7) is the tangency condition (5.5) transformed using the homogeneity property of the function (3.2): $F = \langle F_p, p\rangle + 1$. The corresponding system (5.7) is considered for each branch $\Gamma^{\pm}$. Besides the trivial solutions $p = q^{\pm}$, both systems (5.7) for noncritical values α, ν have a common solution $p = (q^+(z) + q^-(z))/2 = R_z(z)$, which is the solution used in (5.6). The first equation in (5.7) is satisfied by the value $p = R_z$ because of (5.2). When this value is substituted in the tangency condition [the second equality in (5.7)], it is transformed to the first equality. Finally, the last equalities in (5.7) are checked using the differential consequences $\langle q^+ - q^-, r_j\rangle = 0$ of the equality $S^+(z) - S^-(z) = 0$, $z \in B$.

The fact that a common value $p(z)$, $z \in B$, exists for both branches of the equivocal surface indicates that, first, the gradient of the value of the game is continuously continuable to the set $\Gamma^+ + B + \Gamma^-$ in the secondary region and, second, the branches Γ^+ and Γ^- are tangent on the set B to one another and to the surface Γ_1. Figure 7 accordingly shows the tangent surfaces.

The vector $p = R_z(z)$ is continuous in the parameters $(\alpha, \nu) \in \Pi$. For the critical values $(\alpha, \nu) \in \Pi_*$ we have $R_z(z) = p^*(1, 0, 0, 0)$. The function F_p is nonsmooth at the point p^*, its gradient F_p used in the second equation (5.7) does not exist. As a function of the parameters (α, ν), the vector $F_p(R_z(z))$ is discontinuous on the curve Π_*. Its limiting values for $(\alpha, \nu) \to \Pi_*$ in the regions Π_1 and Π_2 are respectively given by

$$\Pi_1: F_p = (1, 0 - \nu, 0), \ \Pi_2: F_p = (1, 0, \nu, 0) \tag{5.8}$$

In general, the even components of the vector $F_p(R_z(z))$ are zero, and the first component is positive in the entire region Π; the third component changes its sign on Π_*.

Three equations of system (5.7) are satisfied by the value p^*, and the second equation holds in the following generalized sense: both vectors (5.8) are contained in the cone of supporting normals to the level surface $F(p) = 0$ at the point p^* (the subdifferential) and are orthogonal to both vectors $p^* - q^{\pm}$.

Moreover, in the critical case $(\alpha, \nu) \in \Pi_*$ the coefficient of $p - q$ in the second equation in (5.6) is meaningless. A rigorous analysis of the Cauchy problem (5.6) requires investigating the asymptotic behaviour of its solution, which is not done here. Below we assume that the parameters α, ν take noncritical values.

To construct the surfaces $\Gamma^{\pm}$, we need to draw the trajectories of the system (5.6) from all points of B; the z-components of the solutions form the surfaces required.

Self-similarity (Section 2) shows that the collection of the trajectories (5.6) can be constructed by simple enumeration of one trajectory (any one). Indeed, Eqs (5.6) and the function $R_z(z)$ are invariant under a change of coordinates and time $z = \lambda z'$, $t = \lambda t'$, $\lambda > 0$. Because of rotational symmetry, we can choose the local coordinates x, y so as to reduce any point of the set B to the form

$$z^{\circ} = \lambda z_B, \; z_B = (1, 0, \rho_B, 0), \; \lambda > 0 \tag{5.9}$$

where ρ_B is defined in (5.4). Thus, in (5.6) it suffices to integrate numerically one standard trajectory originating from the point (5.9), e.g. for $\lambda = 1$. The standard trajectories for the regions Π_1 and Π_2 differ by the sign of the third component of the vector F_p in (5.6) at the initial instant of time. This means that on the initial section of the trajectory (in reverse time) the point E of the region Π_1 (Π_2) moves toward the vertex (away from the vertex) of the cone. The player P always moves away from the vertex. The corresponding sections of the trajectory are shown in Fig. 2; the index of the region Π_i is shown next to the curve.

The equivocal curve in self-similar variables is shown in Figs 4 and 5. In reverse time, integration is from point B to Q. Point Q is reached in an infinite time; the primary trajectories cross the curve BQ at a nonzero angle. The last assertion is based on the following property of system (5.6): the Langrangian manifold $\Sigma = \{(z, p)\colon\ p = q(z),\ z \in Z\}$ contains an attracting submanifold of the system which is reached by the solution in infinite time. In other words, for large (reverse) times, the equivocal motion is close to primary motion. This property is established by analysing the variational equation for the vector $w = p - q$, which is obtained by expanding the right-hand sides of (5.6) in powers of w.

The curve BQ is tangent to the ray OB at the point B, and $\rho^{\bullet} < 0$ (in reverse time) for all parameter values. The effect shown in Fig. 2 in Cartesian variables corresponds to a discontinuity of $\rho^{\bullet}$ on the critical curve with the following values of the limits: Π_1: $\rho^{\bullet} = -(\rho_B + \nu) < 0$ Π_2: $\rho^{\bullet} = -(\rho_B - \nu) < 0$.

6. SECONDARY SOLUTION. COMPLETION OF OPTIMAL SYNTHESIS

In direct time, the trajectories are tangent to the equivocal surface Γ. To construct the trajectories, we need to draw the solutions of the regular system (3.6) in reverse time from the points of Γ. The initial values of the conjugate variables are the values of the vector $p(z)$, $z \in \Gamma$, obtained by integrating the system of singular characteristics (5.5). The trajectories issued from Γ in reverse time fill some (secondary) region Z_2.

For the parameter region Π_1, the set Z_2 is the curvilinear triangle OBQ (Fig. 4). The segment OB is the secondary trajectory tangent at the point B to both branches Γ^+ and Γ^- of the equivocal curve (one branch is shown; the other is in the symmetrical half of the figure). The secondary trajectories in different halves of the figure intersect at the points of the open segment OQ, which are reached at different times, i.e. OQ is a scattering line for $(\alpha, \nu) \in \Pi_1$.

This and a number of other assertions of this section rely on the analysis of some global properties of the solutions of the Cauchy problem (5.6) in Cartesian and self-similar variables.

Thus, in the region Π_1 of the parameters (α, ν), the entire phase space is partitioned into two regions Z_1, Z_2, whose boundaries include three singular lines—two scattering lines and one equivocal line.

For $(\alpha, \nu) \in \Pi_2$, a third region Z_3 is also formed: $Z = Z_1 + Z_2 + Z_3$ (Fig. 5). In this region, the optimal result is determined only by the position of the player P and is equal to the time of his motion to the vertex O, where capture occurs, assuming optimal behaviour of player E.

To obtain a more detailed description of the synthesis for Π_1, let us first consider a qualitative comparison of the optimal pursuit strategies in regions Π_1 and Π_2 for the boundary values $\alpha = 0, \pi$. For the region Π_1, the motion of the players is similar to pursuit in a plane ($\alpha = \pi$), when the point E evades P and the existence of the vertex is not an obstacle to this evasion, merely "deforming" the trajectories to a certain extent.

Now let $\alpha = 0$, i.e. the space K is a ray with the end point O. If the vertex O and the player E are initially on different sides of the point P, the player P moves with maximum velocity toward E until it is captured. A similar optimal pursuit is realized when E and O are on the same side of P, but the distance ratio of the players from the vertex satisfies the condition $r/R \leqslant \nu$. Then capture occurs not later than the arrival of the players at the point O. If $r/R < \nu$, then obviously capture occurs at the vertex: the player P moves with maximum velocity to the vertex and player E, acting ambiguously, will reach the vertex at the required time $T = R$. Here the capture time is independent of r.

Pursuit of the cone for $(\alpha, \nu) \in \Pi_2$ is similar to the situation on a ray. The region Z_3, which is defined by the same inequality $\rho = r/R < \nu$, corresponds to a position of the players such that the point P manages to "crowd" the player E toward the vertex, whose existence restricts the manoeuvrability of E.

In the previous section, we have described the jumplike variation of the standard equivocal trajectory of the player E (the sign changes of the variable $z_3^{\bullet}$) when the parameters cross from region Π_1 to Π_2. Singular equivocal trajectories for the region Π_2 are such that the secondary trajectories issuing in reverse time from the branches Γ^+, Γ^- do not intersect one another and are extended to an infinite time interval. The collection of secondary z-trajectories issuing from the solutions of system (5.6) given condition (5.9) tend for $\lambda \to 0$ to some surface Γ_3, which defines the region Z_3. The surface Γ_3 consists of all trajectories along which the points P and E, moving with maximum velocity toward the vertex, reach the vertex simultaneously. In self-similar variables, Γ_3 is an arc of the circle $\rho = \nu$, $|\varphi| \leqslant \alpha$, whose points are reached in infinite time by the trajectories starting on the curve BQ.

Inside the region Z_3, the optimal result depends only on the position of the pursuer: $V = V(z_1, z_2) = R$, $V_{z_3} = 0$. On Γ_3, the function $V(z)$ is nonsmooth (and Γ_3 is thus a singular surface), but both conditions (3.1) are satisfied as equalities. This means that players P and E have pure positional controls on Γ_3.

Equivocal and secondary regular trajectories, in particular, those shown in Figs 4 and 5, were constructed by numerical integration. The convexity properties of the standard trajectories of the players ensuring single-valued filling of the secondary region by regular trajectories are also partially justified by the analysis of numerical results.

Thus, the necessary optimality conditions (3.1)–(3.7), (5.5) and their consequences (5.1) and (5.6) have led to single-valued constructions in the entire game space. A complete proof of the optimality of the proposed synthesis requires further study, in particular, an investigation of the properties of the solutions of the Cauchy problem (5.6).

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TWO PROBLEMS OF ENCOUNTER UNDER CONDITIONS OF UNCERTAINTY†

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Two problems of the encounter of several controlled objects described by nonlinear differential inclusions, with controls in the right-hand side are considered. Necessary conditions of optimality are obtained in the form of a maximum principle. Previously such problems have been considered for the one-dimensional case [1] and for the multidimensional linear case.‡

1. LET R^n be the n-dimensional real Euclidean space with the norm $\|x\| = (x_1^2 + \ldots + x_n^2)^{1/2}$, $x = (x_1, \ldots, x_n) \in R^n$. We denote by $\operatorname{conv}(R)$ the space of all nonempty compact and convex subsets in R^n. The metric $h(A, B)$ between the sets A, B in $\operatorname{conv}(R^n)$ is defined by the formula

$$h(A, B) = \min \{r \geqslant 0 \mid A \subset B + S_r(0),\ B \subset A + S_r(0)\},$$

where $S_r(a)$ is the sphere in R^n with the radius $z > 0$ centred at the point $a \in R^n$.

Denote by $\operatorname{cc}(R^n)$ the space of all nonempty compact subsets of the space $\operatorname{conv}(R^n)$ with the metric

$$\delta(A, B) = \max \{\max_{a \in A} \min_{b \in B} h(a, b), \max_{b \in B} \min_{a \in A} h(a, b)\}$$

† *Prikl. Mat. Mekh.* Vol. 55, No. 5, pp. 752–758, 1991.

‡ RADZHEF M. S., Investigation of one problem of optimal control of M-objects with multivalued trajectories, Odessa State University, Odessa, 1983. Unpublished manuscript, UkrNIINTI 30.01.84, No. 137–Uk84.